The Seventh Roots of Unity

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To evaluate the seventh roots of unity in radicals, notice that they satisfy the equation x^7-1 = 0. The number 1 is a root of this equation, since $1^7 = 1$. Therefore, x - 1 is a factor of x^7- 1. If you perform the division, the result is:

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

One way of solving this equation, and indeed of $x^7 - 1 = 0$, is to use the cis notation. Note that the seven roots are given by $e^{2\pi ni/7}$, which can also be expressed as cos $(2\pi n/7)+i$ sin$(2\pi n/7)$, or to abbreviate, cis $(2\pi n/7)$. This is not really solving by radicals, as it uses trigonometric functions. But it does show that if x is a root of the equation, then so is $1/x$, for the inverse of cis $(2\pi n/7)$ is cis $((2\pi(7-n)/7)$. Another way to see this is that the equation polynomial is a palindrome. Replacing x by $1/x$ in the equation causes the coefficients to go in reverse order, meaning you get the same polynomial again. Note also that if r is a root of the equation, that

r + 1/r
= cis $(2\pi n/7)$ + cis $((2\pi(-n)/7)$
= cos $(2\pi n/7)$ + I sin $(2\pi n/7)$ + cos $(2\pi n/7)$ - I sin $(2\pi n/7)$
= 2 cos $(2\pi n/7)$

which is double the real part of one of the seventh roots of unity. So the idea is to compute $r + 1/r$, and then get the imaginary part as $\sqrt{1-r^2}$.

If r and 1/r are roots of a quadratic polynomial, then that polynomial is $x^2 - (r+1/r)x + 1 = 0$, since the product of the two roots is 1. The roots exist in three pairs, each inverse (and also complex conjugate) of each other. This suggests expressing the polynomial as:

$$(x^2 + Ax + 1)(x^2 + Bx + 1)(x^2 + Cx + 1) = 0$$

Now expand this polynomial, and then compare to the equation above. This results in a system of equations.

$A + B + C = 1$
$AB + AC + BC + 3 = 1$
$ABC + 2(A+B+C) = 1$

From these equations, we conclude that

$A + B + C = 1$
$AB + AC + BC = -2$
$ABC = -1$

This in turn means that A, B, and C satisfy the equation:

$y^3 - y^2 - 2y + 1 = 0$

This cubic can be solved using the cubic formula. It is going to give real roots, but the formula will express them with complex numbers. That's how it happens. We cannot express them in real radicals.

To solve the cubic, first replace y with $x + 1/3$. This gets rid of the second term, resulting in the equation:

$$x^3 - \frac{7}{3}x + \frac{7}{27} = 0$$

The formula is, where $p = -7/3$ and $q = 7/27$:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
$$= \sqrt[3]{\frac{-7 + 21i\sqrt{3}}{54}} + \sqrt[3]{\frac{-7 - 21i\sqrt{3}}{54}}$$

This gives x. To get y, remember we made the substitution, and note that $54/3^3 = 2$:

$$y = \frac{1 + \sqrt[3]{\frac{-7 + 21i\sqrt{3}}{2}} + \sqrt[3]{\frac{-7 - 21i\sqrt{3}}{2}}}{3}$$

This is only one of the roots of the cubic in y, and I have used an ambiguous notation - the cube root. There are three possible cubic roots. What do I mean? In this expression I mean take the polar coordinates of what's under the cube root sign, and take one-third of the angle. If I choose the second radical to be the complex conjugate of the first, the resulting y is a real number, namely $2 \cos (2\pi/7)$. One-half of this number is the real part of one of the seventh roots of unity. To express the other two roots, I would have to insert factors of ω and ω^2, and ω^2 and ω, to the cube roots above, where ω is one of the complex cube roots of unity.

To get the imaginary part of the seventh roots of unity, use the fact that the modulus, or absolute value, of the roots is 1. This means that the imaginary part z is:

$$z = \sqrt{1 - y^2} = \sqrt{\frac{8 + \sqrt[3]{\frac{-7 + 21i\sqrt{3}}{2}} + \sqrt[3]{\frac{-7 - 21i\sqrt{3}}{2}} + \sqrt[3]{\frac{-637 - 147i\sqrt{3}}{2}} + \sqrt[3]{\frac{-637 + 147i\sqrt{3}}{2}}}{54}}$$

So an expression for the 7th roots of unity is:

$$\frac{1}{3}\left(\frac{1+\sqrt[3]{\frac{-7+21i\sqrt{3}}{2}}+\sqrt[3]{\frac{-7-21i\sqrt{3}}{2}}}{2}+i\sqrt{1-\frac{8+\sqrt[3]{\frac{-7+21i\sqrt{3}}{2}}+\sqrt[3]{\frac{-7-21i\sqrt{3}}{2}}+\sqrt[3]{\frac{-637-147i\sqrt{3}}{2}}+\sqrt[3]{\frac{-637+147i\sqrt{3}}{2}}}{2}}\right)$$

The real part of this root is a solution of a cubic equation, but I don't believe that the imaginary part is - it is the root of a sixth degree equation. A similar phenomenon exists for the fifth roots of unity.

To get all six roots, one takes all six possible combinations of the sign before the I in the second line and the ωs and ω^2s as coefficients of the cube roots. Note that once you make a choice of ωs and ω^2s in the real part, one must keep the same choice in the imaginary part.

To see what happened when I evaluated these roots, note that I solved a cubic equation. Adding on to the rational numbers (**Q**) the roots of this equation results in a degree 3 extension (not degree 6, since when one adds one root, the other roots also appear in the field). This is the splitting field of $y^3 - y^2 - 2y + 1 = 0$ over **Q**. This got me the real part of the complex roots. I then added *i* to this field, creating a degree 6 extension. This extension, obtained by adding the root of $y^3 - y^2 - 2y + 1 = 0$ and *i* to **Q**, is the splitting field F of $x^7 - 1 = 0$ over **Q**. The Galois group of all automorphisms of F over **Q** that fixes **Q** is **Z**/6**Z**, the integers mod 6. In terms of the Galois group, I went from I to **Z**/3**Z to Z**/6**Z.**

What I wonder is, what if you go up through **Z**/2**Z** instead? In that case, you have to come up with some quadratic equation, and then express the polynomial as a cubic in that field. What I do know is that the number that you have to add to **Q** to get this quadratic extension is $r + r^2 + r^4$, where the exponents are the quadratic residues of 7. This number is $\frac{-1+i\sqrt{7}}{2}$.